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A network approach to welfare

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Abstract

This paper investigates the welfare impact of income redistribution for public goods in networks. First, we measure the impact of income redistribution and show that it affects each consumer only insofar as it affects his neighbourhood. Second, we characterise Pareto-improving income redistributions and relate them to the network structure. Third, in the case of Cobb–Douglas preferences, we establish a new link between two well-known concepts of the comparative statics of income redistribution, the neutrality result and the transfer paradox. Collectively, our findings uncover the importance of the -1 eigenvalue to economic and social policy: it is an indication of how the network structure induces consumers to absorb the impact of income redistribution.

JEL classification: C72, D31, D60, D85, H41.

Keywords: public goods, income redistribution, welfare effect, networks, neutrality, transfer paradox.

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1 Introduction

Market outcomes provide scope for economic policy to raise welfare, and income redistribution is often considered as the benchmark for welfare-motivated policies. For instance, the Second Welfare Theorem for competitive equilibrium, where efficiency is always assured and welfare is only affected in a normative sense through improvements to equity. Research on the private provision of public goods has also focused on welfare implications. Important results by Bergstrom, Blume and Varian (1986) (henceforth BBV) show, firstly, that the public good is under-provided relative to the efficient level and, secondly, that income redistribution among contributors that leaves the set of contributors unchanged is 'neutral'. Neutrality means that contributors adjust their contributions to exactly offset the transfer among contributors that leaves the set of contributors unchanged so that there is no change in consumption.

This paper explores the pattern of welfare impacts due to income redistribution for the private provision of public goods in networks. Public goods are often 'local' in the sense that consumers only benefit from the provision of their direct neighbours. Examples include information acquisition, liquidity in a banking network, and municipal amenities such as lighthouses or farming irrigation systems. In a key contribution, Bramoullé and Kranton (2007) showed that the network context, where local influences are heterogeneous among consumers, is a natural setting to examine the private provision of public goods. Bramoullé, Kranton and D'Amours (2014) investigated the whole range of strategic substitution and identified a threshold of impact related to the lowest eigenvalue of the network. Below the threshold, the uniqueness and stability of a Nash equilibrium hold. Beyond it, multiple Nash equilibria will in general exist, and stability holds only for corner equilibria. Allouch (2015) extended this model to the non-linear case, with a condition on the normality of the public good which follows BBV's approach, and showed that neutrality no longer holds for income redistribution in general networks. This result opens the door to policy interventions that could improve welfare. Allouch (2017) first investigated the benchmark policy of income redistribution between contributors, focusing on preferences that yield affine Engel curves[1] and using a standard utilitarian approach. Other recent and relevant contributions to the network literature include those by: Galeotti, Goyal, Jackson, Vega-Redondo and Yariv (2010); Ghiglino and Goyal (2010); Elliott and Golub (2015); Acemoglu, Malekian and Ozdaglar (2016); Bourlès, Bramoullé and Perez-Richet (2017); Kinateder and Merlino (2017) and López-Pintado (2017).

Our main contributions are threefold. First, we find a property that is key to understanding the impact of income redistribution. More specifically, we show that a transfer affects each consumer only insofar as it affects the consumer's neighbourhood, formed by herself and her neighbours. That is, it is the aggregate transfer to the consumer's neighbourhood, rather than the individual transfer to the consumer, that affects consumption. Hence the policy implications of income redistribution can be derived by focusing just on the classes of 'policy-equivalent' transfers: those with identical corresponding neighbourhood transfers. Next we investigate neutral transfers in general networks, which are policy-equivalent to the null transfer and hence represent the kernel of policy constraints faced by the social planner.

[1] Of which Cobb–Douglas preferences is a special case.

Secondly, given that income redistribution can affect consumption, we characterise Pareto-improving income redistributions. We follow the policy reform literature in the tradition of Dixit (1975), Guesnerie (1977), Weymark (1981), and Ahmad and Stern (1984), who typically aim to design small feasible changes that will improve welfare, when the optimal outcome is not readily available or requires a significant change in the state of the economy. We find two mutually exclusive cases — either there is a Pareto-improving income redistribution or, if not, we can identify the implicit welfare weights of the initial private provision equilibrium. In this context we identify a class of networks where a Pareto improvement always exists for any profile of preferences of consumers. This is very much in the spirit of the BBV neutrality result, which also holds for any profile of preferences of consumers. Additionally, in the case of Cobb–Douglas preferences, we show that the feasibility of Pareto-improving reform turns out to be readily interpreted and easily checked from the network structure. As a consequence, our policy reform analysis leads to a succinct characterisation of the welfare impact of income redistribution.

Thirdly, in the case of Cobb–Douglas preferences, we provide a link between two well-known, but seemingly unrelated, concepts of the comparative statics of income redistribution, the neutrality result and the transfer paradox (Leontief, 1936; Samuelson, 1952; Yano, 1983; Balasko, 2014). In the case of a Pareto improvement, the utility level of the donors must move in the opposite direction to their transfer, and this is called a *weak* transfer paradox. In contrast, a *strong* transfer paradox occurs when the utility levels of both donors and recipients move in the opposite direction to the transfer. In fact, by focusing on transfers that are also eigenvectors, our network approach to welfare shows that the existence of a strong transfer paradox can be related to the neutrality result. More specifically, we show that neutrality corresponds to the point of policy switch between transfers where the utility levels of the donors and the recipients move in the same direction (normal welfare impact), and transfers where the utility levels of the donors and the recipients move in the opposite direction (paradoxical welfare impact).

This paper is the first to show the importance of the -1 eigenvalue to social and economic outcomes, since our findings identify it as a condition for neutral transfers, Pareto improvement, and the policy switch. In interpretation, the -1 eigenvalue is an indication of how consumers, via their neighbourhood, absorb the impact of income redistribution, and hence of the welfare implications. It is not used as a common measure of network analysis in any other fields, including sociology, computer science, and physics. Given that the -1 eigenvalue provides a key to social and economic outcomes, perhaps its relationship to the underlying network structure could usefully be studied alongside classic network statistics such as the the highest, the second, and lately the lowest eigenvalues.

Finally, aggregative games have been a subject of ongoing interest in economics, given their valuable insights in the study of a variety of strategic interactions including public goods provision, rent-seeking contests, and patent races. Our findings for public goods in networks highlight the interplay between an aggregative structure of the game, whereby each agent's payoff depends only on his own action and the sum of his neighbours' actions, and the underlying network structure of the game. In this respect, we think our analysis could also be useful for the nascent literature on aggregative network games; see, for example, Melo (2017) and Parise and Ozdaglar (2017).

The paper is structured as follows. Section 2 sets out the model and Section 3 investigates policy-equivalent transfers and neutrality. Section 4 investigates Pareto-improving transfers, Section 5 provides a new perspective on neutrality and the transfer paradox, and Section 6 discusses the welfare implications of the -1 eigenvalue. Section 7 concludes the paper

2 The model

We consider a society comprising n consumers who occupy the nodes of a fixed network $\mathbf{g}$ of social interactions. Let $\mathbf{G} = [g_{ij}]$ denote the adjacency matrix of the network $\mathbf{g}$, where $g_{ij} = 1$ indicates that consumer $i \neq j$ are neighbours and $g_{ij} = 0$ otherwise. The adjacency matrix of the network, $\mathbf{G}$, is symmetric with non-negative entries and therefore has a complete set of real eigenvalues (not necessarily distinct), denoted by $\lambda_{\max}(\mathbf{G}) = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = \lambda_{\min}(\mathbf{G})$, where $\lambda_{\max}(\mathbf{G})$ is the largest eigenvalue and $\lambda_{\min}(\mathbf{G})$ is the lowest eigenvalue of $\mathbf{G}$. By the Perron–Frobenius Theorem, it holds that $\lambda_{\max}(\mathbf{G}) \geq -\lambda_{\min}(\mathbf{G}) > 0$.

Consumer i's neighbours in the network $\mathbf{g}$ are given by $\mathcal{N}_i$. The preferences of each consumer i are represented by a twice continuously differentiable, strictly increasing, and strictly quasi-concave utility function $u_i(x_i, q_i + Q_{-i})$, where x_i is consumer i's private good consumption, q_i is consumer i's public good provision, and $Q_{-i} = \sum_{j \in \mathcal{N}_i} q_j$ is the sum of public good provisions of consumer i's neighbours in the society. Furthermore, the public good can be produced from the private good via a unit-linear production technology. That is, any non-negative quantity of the private good can be converted into the same quantity of the public good. For simplicity, the prices of the private good and the public good can be normalised to $\mathbf{p} = (p_x, p_Q) = (1, 1)$. Each consumer i faces the utility maximisation problem

$$\max_{x_i, q_i} u_i(x_i, q_i + Q_{-i})$$
$$\text{s.t.} \quad x_i + q_i = w_i \text{ and } q_i \geq 0,$$

where w_i is his income (exogenously fixed). The utility maximisation problem can be represented equivalently as

$$\max_{x_i, Q_i} u_i(x_i, Q_i)$$
$$\text{s.t.} \quad x_i + Q_i = w_i + Q_{-i} \text{ and } Q_i \geq Q_{-i},$$

where consumer i chooses his (local) public good consumption, $Q_i = q_i + Q_{-i}$. Let γ_i be the Engel curve of consumer i. Then consumer i's local public good demand depends on $w_i + Q_{-i}$, each consumer's 'social wealth' (Becker, 1974):

$$Q_i = \max\{\gamma_i(w_i + Q_{-i}), Q_{-i}\},$$

or equivalently,

$$q_i = Q_i - Q_{-i} = \max\{\gamma_i(w_i + Q_{-i}) - Q_{-i}, 0\}.$$

We will assume, throughout the paper, the following network-specific normality assumption, which amounts to both the normality of the private good and a strong normality of the public good:

Definition 2.1. ***Network normality.*** *(Allouch, 2015) For each consumer $i = 1, \dots, n$, the Engel curve γ_i is differentiable and it holds that $1 + \frac{1}{\lambda_{\min}(\mathbf{G})} < \gamma_i'(\cdot) < 1$.*

Theorem 2.2. *(Allouch, 2015) Assume network normality. Then there exists a unique Nash equilibrium in the private provision of public goods on networks.*

3 Income redistribution in networks

Now, we investigate the impact of a social planner's intervention on the private provision of public goods. The social planner aims to achieve socially optimal outcomes by drawing on income redistribution as a policy instrument. Income redistribution takes the form of lump-sum transfers, which are traditionally viewed as a benchmark for other policy instruments. In general, not all consumers will be contributing to public goods. Therefore, at a Nash equilibrium, there will be possibly several components of contributors. We will focus our analysis on just one component S of contributors. For simplicity of notations, by passing to the subnetwork, we will assume $S = N$. Let $\mathbf{q}^* = (q_1^*, \dots, q_n^*)$ be the Nash equilibrium associated with $\mathbf{w} = (w_1, \dots, w_n)$ and let $\mathbf{t} = (t_1, \dots, t_n)$ denote a budget-balanced income transfer, that is,

$$\sum_{i=1}^{n} t_i = 0,$$

where transfers could be a tax $(t_i < 0)$ or a subsidy $(t_i \geq 0)$. Let $\mathbf{q}^t = (q_1^t, \dots, q_n^t)$ be the Nash equilibrium after an income transfer $\mathbf{t}$, that is, the Nash equilibrium corresponding to the income distribution $\mathbf{w} + \mathbf{t} = (w_1 + t_1, \dots, w_n + t_n)^T$.

Like Warr (1983) and BBV, we first focus our analysis on income redistributions that leave the set of contributors unchanged, referring to them as 'relatively small'. Given a vector $\mathbf{u}$, let $\hat{\mathbf{u}} = (\mathbf{I} + \mathbf{G})\mathbf{u}$ denote the corresponding *neighbourhood* vector.

3.1 Neighbourhood transfers and policy equivalence

Now we investigate transfers that lead to the same change in private and public good consumptions. We say that two transfers $\mathbf{t}_1$ and $\mathbf{t}_2$ are *policy-equivalent*, if for each i, it holds that

$$(x_i^{\mathbf{t}_1}, Q_i^{\mathbf{t}_1}) = (x_i^{\mathbf{t}_2}, Q_i^{\mathbf{t}_2}).$$

Proposition 3.1. *Two relatively small transfers $\mathbf{t}_1$ and $\mathbf{t}_2$ are policy-equivalent if and only if their corresponding neighbourhood transfers are identical, that is, $\hat{\mathbf{t}}_1 = \hat{\mathbf{t}}_2$.*

Proposition 3.1 shows that it is the neighbourhood transfer, rather than the transfer itself, that determines the policy impact on consumers. That is, a transfer impacts each consumer's consumption of either the public or private good only insofar as it impacts his neighbourhood. As a consequence, policy implications of income redistribution can be derived from focusing just on the classes of policy-equivalent transfers, rather than all transfers. That said, policy-equivalent transfers, since they reduce the scope of income redistribution, also indicate the policy constraints faced by the social planner due to the network structure. The following example illustrates policy-equivalent transfers.

Example 1. Consider the line network with five consumers. Figure 1 shows two policy-equivalent transfers $\mathbf{t}_1$ and $\mathbf{t}_2$ since they have identical corresponding neighbourhood transfers. It is worth noting that the two transfers $\mathbf{t}_1$ and $\mathbf{t}_2$ seem unrelated since the consumers involved with transfer $\mathbf{t}_1$ are not adjacent to consumers involved with transfer $\mathbf{t}_2$.

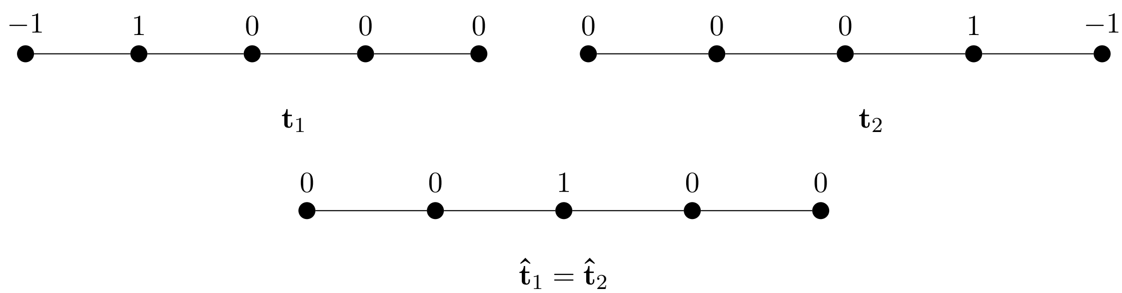


Figure 1: Policy-equivalent transfers

3.2 Neutral transfers

An important class of policy-equivalent transfers, which represents the kernel of policy constraints faced by the social planner, are neutral transfers. Neutral transfers are policy-equivalent to the null transfer and therefore have no impact on consumers in terms of their consumption of either the public or private good. By virtue of policy equivalence, neutral transfers are characterised by having a null neighbourhood transfer.

Corollary 3.1. *A relatively small transfer* $\mathbf{t}$ *is neutral if and only if* $\hat{\mathbf{t}} = \mathbf{0}$.

An example of network structures where neutral transfers always occur are those with a *neighbourhood homogeneous* subset of consumers. We say a subset of consumers H is neighbourhood homogeneous, if for any i, j in H, it holds that $i \cup \mathcal{N}_i = j \cup \mathcal{N}_j$. Observe that the neighbourhood of each consumer $i \in N$ either includes all consumers or no consumer in H. This implies that any transfer among consumers in H is neutral since it induces a null corresponding neighbourhood transfer. In particular, the neutrality result of BBV for pure public goods, or equivalently for complete networks, holds since all subsets are neighbourhood homogeneous.

In general, neutral transfers occur in network structures both with and without a neighbourhood homogeneous subset of consumers. We illustrate this in the following example.

Example 2. Figure 2 shows a neutral transfer for a network with a neighbourhood homogeneous subset of consumers. Meanwhile, Figure 3 shows a neutral transfer for a network without a neighbourhood homogeneous subset of consumers.

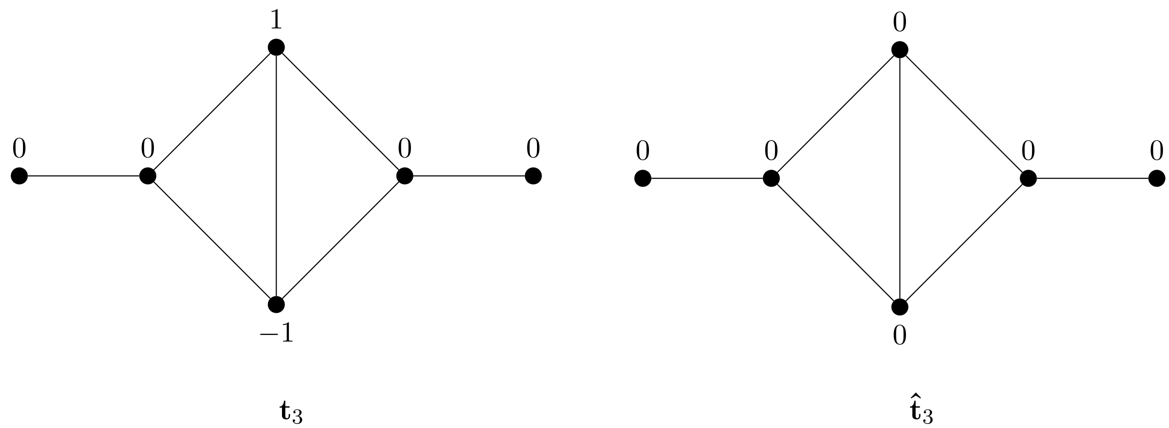


Figure 2: A network with a neighbourhood homogeneous subset

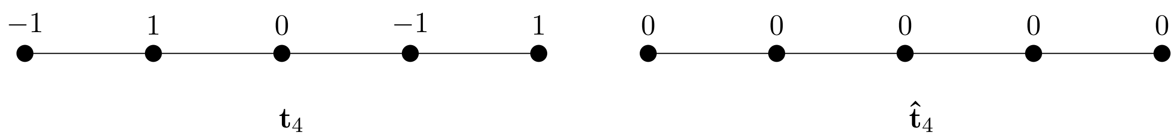


Figure 3: A network without a neighbourhood homogeneous subset

3.3 Dimension of neutral transfers

Neutral transfers not only have no impact on consumption, but also, by virtue of policy equivalence, restrict the policy outcomes that income redistribution can achieve. A natural question which follows is the dimension of neutral transfers in general networks. To answer this question, the following characterisation of neutral transfers turns out to be useful.

Proposition 3.2. *Neutral transfers are the budget-balanced eigenvectors to the -1 eigenvalue.*

The link between the budget balance requirement of a transfer and the vector $\mathbf{1}$ highlights the role of the concept of *main eigenvalues* in our analysis. A main eigenvalue is an eigenvalue that has an eigenvector not orthogonal to the vector $\mathbf{1}$ (Cvetković, 1970).[2] The distinct main eigenvalues of $\mathbf{G}$ form the main part of the spectrum, denoted by $\mathcal{M}$ (Harary and Schwenk, 1979). Let k denote the multiplicity of the -1 eigenvalue outside the main part of the spectrum. That is,

$$k = \begin{cases} \psi_{\mathbf{G}}(-1) - 1 & \text{if } -1 \in \mathcal{M}, \\ \psi_{\mathbf{G}}(-1) & \text{if } -1 \notin \mathcal{M}, \end{cases}$$

[2] By the Perron–Frobenius Theorem, the maximum eigenvalue of $\mathbf{G}$ has an associated eigenvector with all its entries positive and, therefore, is a main eigenvalue.

where $\psi_{\mathbf{G}}(-1)$ denotes the multiplicity of the -1 eigenvalue in the spectrum of $\mathbf{G}$.

Proposition 3.3. *There are k linearly independent neutral transfers.*

The multiplicity of the -1 eigenvalue outside the main part of the spectrum, k, captures the policy constraints faced by the social planner due to the network structure. Roughly speaking, the higher is k, the fewer policy outcomes that income redistribution can achieve. In particular, the neutrality of BBV occurs as a limiting case, since only the complete network reaches the highest possible k of $n-1$. More generally, if the network has a neighbourhood homogeneous subset H, then a lower bound of k is $|H|-1$, which accounts for the number of linearly independent, binary (between two consumers), and neutral transfers in H. Nonetheless, as we have shown above, k also accounts for neutral transfers without a neighbourhood homogeneous subset of consumers.

4 Pareto-improving transfers

Given that income redistribution can affect consumption, let us further examine the potential welfare impact. More specifically, we investigate whether it is possible to achieve a Pareto improvement with income redistribution. We follow the policy reform literature in the tradition of Dixit (1975), Guesnerie (1977), Weymark (1981), and Ahmad and Stern (1984) and focus our analysis on local transfers, which induce infinitesimal income redistributions. The idea is that implementing a Pareto-optimal outcome is typically unachievable because it requires a significant change from the existing state of the economy. More realistic are local transfers that are Pareto-improving and equilibrium-preserving. Let $\mathbf{v}^* = (u_i(x_i^*, Q_i^*))_{i\in N}$ denote the vector of utilities of all consumers at the initial equilibrium, $\mathbf{v}^{\mathbf{t}} = (u_i(x_i^{\mathbf{t}}, Q_i^{\mathbf{t}}))_{i\in N}$ denote the vector of utilities after local transfer $\mathbf{t}$, and $\Delta\mathbf{v}(\mathbf{t}) = \mathbf{v}^{\mathbf{t}} - \mathbf{v}^*$.

Proposition 4.1. *There are two mutually exclusive possibilities, (a) and (b):*

(a) There exists a Pareto-improving local transfer.

(b) There exists an $\mathbf{r} \in \mathbb{R}^n_{++}$ for which the initial private provision equilibrium is welfare-optimal. That is, it holds that $\mathbf{r}\cdot\Delta\mathbf{v}(\mathbf{t}) = 0$ for any local transfer $\mathbf{t}$.

Proposition 4.1 shows that either a local Pareto-improving redistribution can be found or the initial private provision equilibrium is an optimum amongst the private provision equilibria achieved by local income redistributions. In the case of no possible Pareto improvement, the $\mathbf{r} \in \mathbb{R}^n_{++}$, also known as *Motzkin weights*,[3] represents the implicit welfare weights at the initial equilibrium.

Proposition 4.2. *If $-1 \in \mathcal{M}$ then there exists a Pareto improvement.*

This proposition is very much in the spirit of the BBV neutrality, since it holds for any profile of preferences. In interpretation, if -1 is a main eigenvalue, then a corresponding eigenvector is a

[3]The name *Motzkin weights* originates from the application of Motzkin's Theorem of the Alternative to find Pareto-improving income redistributions in Guesnerie (1977).

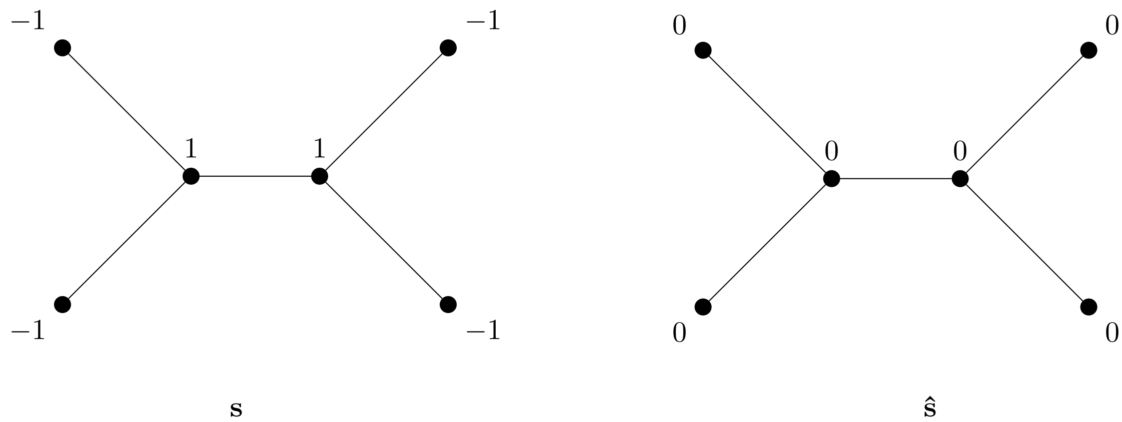


Figure 4: A Pareto-improving tax and subsidy scheme

budget-unbalanced tax and subsidy scheme that is policy neutral since no consumer's neighbourhood is affected. In this regard, the social planner can use the tax and subsidy scheme to create tax revenue which can be redistributed for Pareto improvement. The following example illustrates this.

Example 3. Consider the network in Figure 4, with two core consumers and four periphery consumers. We can observe that the tax and subsidy scheme $\mathbf{s}$ such that $s_p = -1$ for periphery consumers and $s_c = 1$ for core consumers is budget unbalanced but policy neutral since its impact is null in each consumer's neighbourhood. Note that $\mathbf{s}$ is an eigenvector to the main eigenvalue -1. Therefore, the social planner could use the tax and subsidy scheme $\mathbf{s}$ as follows: take 1 unit of income from each of the four periphery consumers then subsidise the two core consumers with 1 unit of income each. This is policy neutral and creates a budget surplus of 2 units of income which can be redistributed for Pareto improvement.

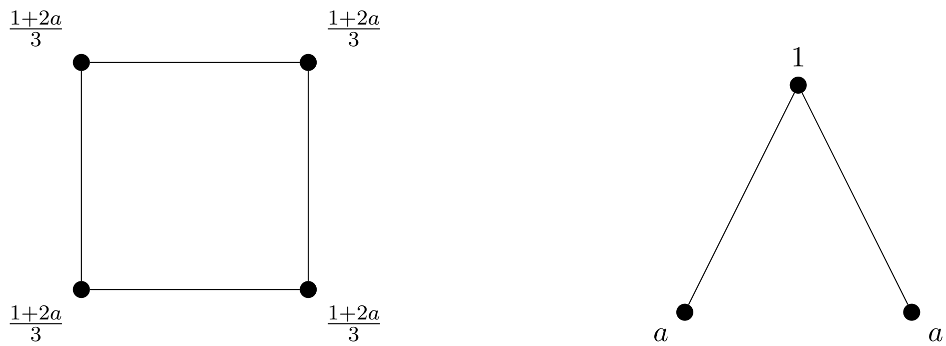
4.1 Cobb–Douglas preferences

In the case of Cobb–Douglas preferences we can further relate the feasibility of Pareto-improving reform to the network structure. Note that since the indirect utility for Cobb–Douglas preferences is linear, we can simply extend our result in Proposition 4.1 for local transfers to relatively small transfers that leave the set of contributors unchanged.

Proposition 4.3. *Assume Cobb–Douglas preferences. Then there are two mutually exclusive possibilities, (a) and (b):*

(a) There exists a Pareto-improving relatively small transfer.

(b) There exists an $\mathbf{r} \in \mathbb{R}^n_{++}$ such that $\hat{\mathbf{r}} = \mathbf{1} + a\mathbf{d}$, where $\mathbf{d}$ denotes degree.



(a) A regular network

(b) A star network

Figure 5: Motzkin weights

In the case of Cobb–Douglas preferences, condition (b) can be interpreted as follows. For each consumer i, the welfare impact in his neighbourhood of an additional unit of tax and subsidy $1 + ad_i$ is equal to the social welfare benefit in his neighbourhood $\hat{r}_i$. This rules out the possibility that the social planner can generate tax revenues that can be redistributed for Pareto improvement. Condition (b) provides a simple method based on the network structure to check whether Pareto-improving transfers exist or not.

Corollary 4.1. *Assume Cobb–Douglas preferences. If the network is regular or star, then there exists no Pareto improvement.*

Indeed, it is easy to check condition (b) holds in the two canonical network structures, regular and star. For a regular network, consider $\mathbf{r} = \frac{1+ad}{1+d}\mathbf{1}$, where d is each consumer's degree. For a star network, consider $\mathbf{r}$ such that $r_c = 1$ for the central consumer and $r_p = a$ for the periphery consumers. The following example illustrates this.

Example 4. Consider the regular network with four consumers in Figure 5a. We can observe that the vector $\mathbf{r} = (\frac{1+2a}{3}, \frac{1+2a}{3}, \frac{1+2a}{3}, \frac{1+2a}{3})$ satisfies condition (b). Consider the star network with three consumers in Figure 5b. Let us designate the central consumer as 1 and the two periphery consumers as 2 and 3. We can observe that the vector $\mathbf{r} = (1, a, a)$ satisfies condition (b).

5 Neutrality and the strong transfer paradox

Now we show that a network approach to welfare, which accounts for heterogeneity of local interactions, provides a link between two key concepts of the comparative statics of income redistribution, the neutrality result and the transfer paradox. Indeed, observe that when a Pareto-improving income redistribution exists, it holds that the donors' utility level must move in the opposite direction to the transfer. In fact, this is an example of a *weak transfer paradox*. On the other hand, a *strong transfer paradox* corresponds to the case where the utility levels of both the donors and recipients move in the opposite direction to the transfer — that is, those who give are better off and those who receive are worse off.

The possibility of a Pareto improvement and transfer paradox may seem surprising. As shown in the utility maximisation, the change in utility of each consumer is an increasing function of their social wealth, $w_i + Q_{-i}$. So there will be two effects on a consumer's utility after a transfer. As a direct effect, the endowment w_i changes by t_i. There will also be an indirect effect on the consumer's social wealth because his neighbours' public good provision Q_{-i} changes in the new equilibrium. Note that the direct effect always works in the direction of the normal or non-paradoxical outcome and that a quite subtle change in the neighbours' public good provision is needed to generate the paradoxes. In the following, we provide an example of a strong transfer paradox.

Example 5. Consider again the star network with three consumers in Figure 5b. Let us designate the central consumer as 1 and the two periphery consumers as 2 and 3. Consider a transfer $\mathbf{t} = (t_1, t_2, t_3)$. Since the transfer is budget-balanced, let $t_1 = -t_2 - t_3$. Then we can write the utility changes of the transfer as

$$\begin{bmatrix} \Delta v_1(\mathbf{t}) \\ \Delta v_2(\mathbf{t}) \\ \Delta v_3(\mathbf{t}) \end{bmatrix} = \kappa_s \begin{bmatrix} a(t_2 + t_3) \\ -a^2 t_2 - (1 - a^2) t_3 \\ -a^2 t_3 - (1 - a^2) t_2 \end{bmatrix},$$

where $\kappa_s > 0$. Observe that if transfers to the periphery consumers are positive, then this *increases* the utility of the central consumer and decreases the utility of the periphery consumers. For each consumer, transfers to themselves have a negative effect on their utility (for consumer 1, this effect can be written as $-at_1$). In fact, this is a *strong transfer paradox* because utility levels move in the opposite direction to the transfer for all consumers — those who give and receive transfers.

Now we show that our welfare analysis, including the existence of a strong transfer paradox, provides a new perspective on the neutrality result. To do so, we will focus our attention on the special case where the transfers are also eigenvectors,[4] given their clear welfare impact.

Proposition 5.1. *Assume Cobb–Douglas preferences. If the transfer* $\mathbf{t}$ *is an eigenvector to the eigenvalue* λ*, then, for some* $\kappa_\lambda > 0$*, it holds that*

$$\Delta \mathbf{v}(\mathbf{t}) = \kappa_\lambda \ (\lambda + 1) \ \mathbf{t} = \kappa_\lambda \ \hat{\mathbf{t}}.$$

Proposition 5.1 shows that if the transfer is also an eigenvector, then its welfare impact is proportional to the corresponding neighbourhood transfer. In addition, it holds that: (i) if the eigenvalue is greater than -1, then the utility levels of both the donors and the recipients move in the same direction as the transfer, which corresponds to the normal welfare impact of the transfer; (ii) if the eigenvalue is equal to -1, the welfare impact is null, which corresponds to neutrality; and (iii) if the eigenvalue is smaller than -1, then the utility levels of both the donors and the recipients move in the opposite direction to the transfer, which corresponds to the paradoxical welfare impact of the transfer (a strong transfer paradox). Therefore, the welfare impact changes

[4] From the definition of main eigenvalues, it can be checked that there are $n - |\mathcal{M}|$ linearly independent transfers that are also eigenvectors.

from one direction to the other depending on the eigenvalue, and the point at which the direction switches, -1, is the point of policy neutrality. In interpretation, the welfare impact can be thought of as a continuous policy function that changes sign, so, at some point it must equal zero, which is the point of policy neutrality. The following example illustrates a network structure where the policy switch occurs.

Example 6. Consider the network shown in Figure 6. Since the transfers we will consider are also eigenvectors, in view of Proposition 5.1, we can measure the welfare impact $\Delta\mathbf{v}(\mathbf{t})$ of the transfer $\mathbf{t}$ by the corresponding neighbourhood transfers $\hat{\mathbf{t}}$. The welfare impact of the transfer $\mathbf{t}_1 = (1, 1, 0, 0, -1, -1)$ is $\hat{\mathbf{t}}_1 = \mathbf{t}_1$, which is a normal welfare impact. The welfare impact of the transfer $\mathbf{t}_2 = (0, 0, 1, -1, 0, 0)$ is $\hat{\mathbf{t}}_2 = 0$, which is neutral. Finally, the welfare impact of the transfer $\mathbf{t}_3 = (1, -1, 0, 0, 1, -1)$ is $\hat{\mathbf{t}}_3 = -\mathbf{t}_3$, which is a paradoxical welfare impact (strong transfer paradox).

Proposition 5.2. *Assume Cobb–Douglas preferences. If the network is regular but not complete, then there exists a transfer — that is also an eigenvector — with a normal welfare impact and a transfer — that is also an eigenvector— with paradoxical welfare impact.*

Corollary 5.1. *The policy switch always occurs in regular but not complete networks.*

Example 7. Consider again the regular network with four consumers in Figure 5a. Figure 7 shows a transfer with a normal welfare impact. Meanwhile, Figure 8 shows a transfer with a paradoxical welfare impact.

6 Welfare implications of the -1 eigenvalue

We have identified the -1 eigenvalue of the network as the condition and a measure for neutral transfers, a sufficient condition for Pareto improvement, and a policy switch point. In addition, given any profile of preferences, its eigenvectors are either neutral transfers with no impact on consumption, if budget-balanced, or tax and subsidy schemes to create tax revenue which can be redistributed for Pareto improvement, if budget-unbalanced. The -1 eigenvalue is also an indication of how consumers, via their neighbourhood, act on, cancel, or counteract income redistribution. Hence the importance and the policy relevance of the -1 eigenvalue.

We are not aware of any mathematical or graph theory research on the relationship between the network structure and the -1 eigenvalue — with the exception of Aouchiche, Caporossi and Hansen (2013), which provides an example of networks having a neighbourhood homogeneous subset of two consumers. The -1 eigenvalue is not a common measure of network analysis in any other fields, including sociology, computer science, and physics. Looking at tables of networks of at most five nodes in Cvetković, Rowlinson and Simić (1997) and of six nodes in Cvetković and Petrić (1984), we notice that the -1 eigenvalue does occur frequently. More precisely, it occurs in more than half of the listed networks, sometimes with multiplicity. In view of the heterogeneity of network structures of at most six nodes, we conjecture that this pattern is not specific to them and

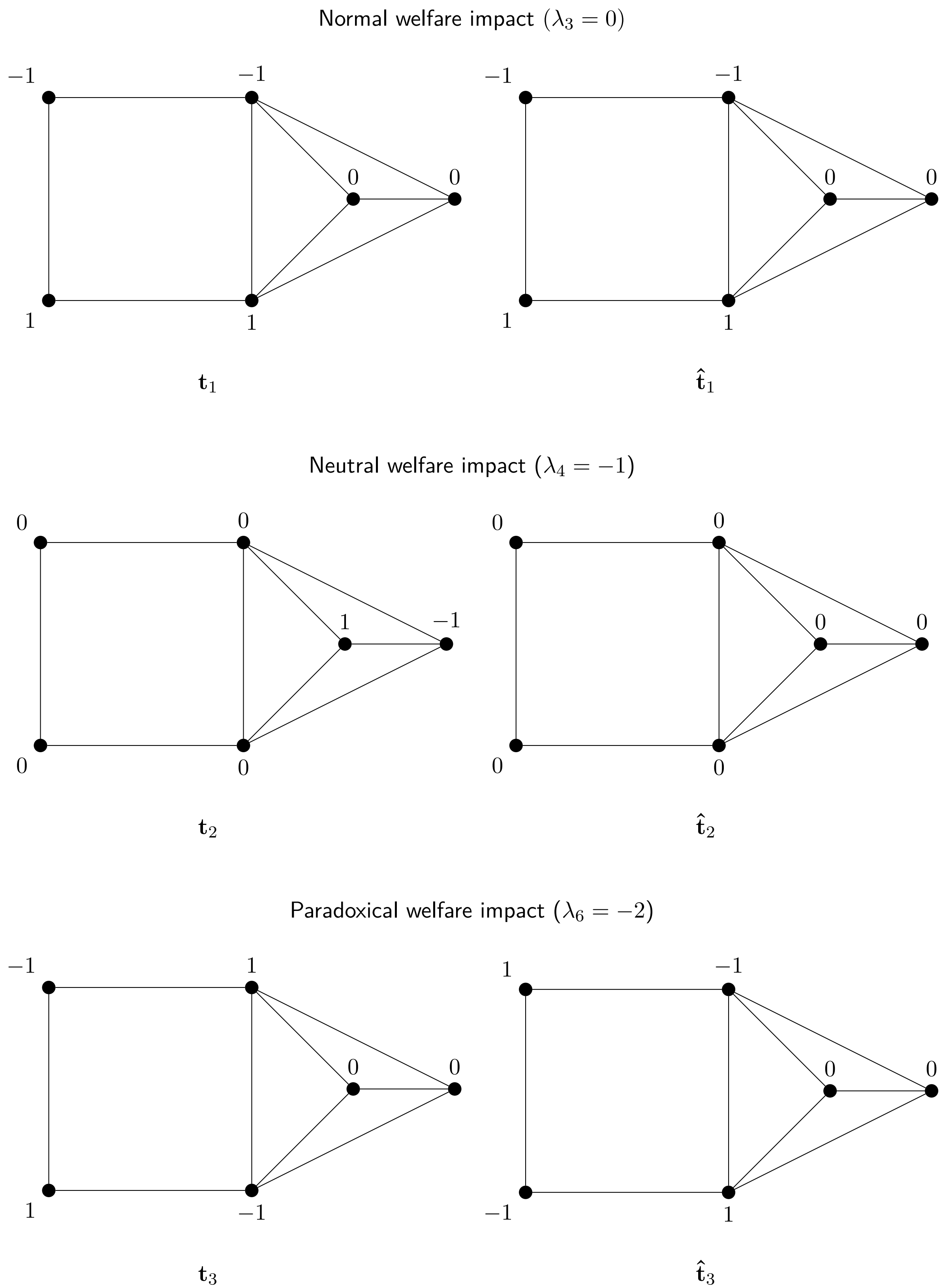


Figure 6: A network with six consumers

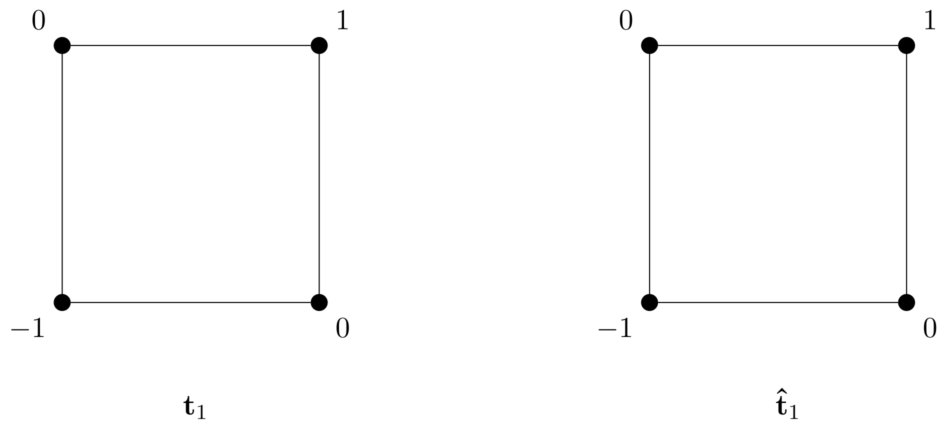


Figure 7: Normal welfare impact

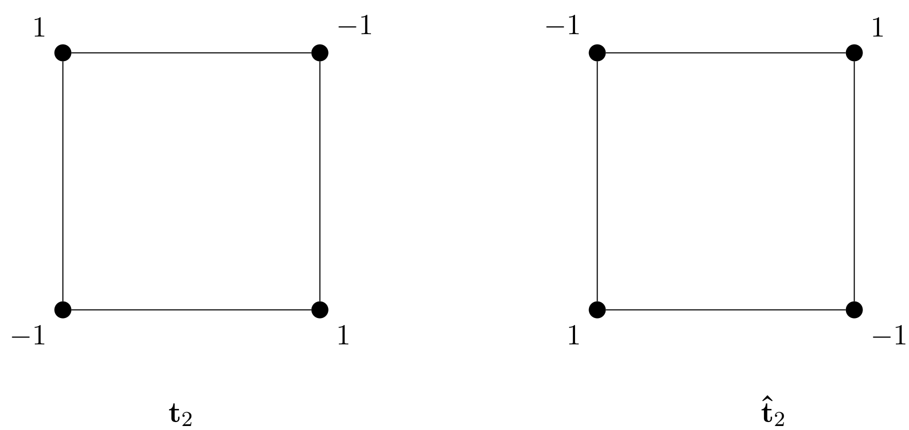


Figure 8: Paradoxical welfare impact

is likely to occur in larger networks. Given the policy relevance of the -1 eigenvalue, perhaps its presence in the spectrum could be usefully studied alongside other important network measures.

7 Conclusion

The welfare implications of income redistribution have featured prominently in several important strands of the literature. In this paper, we have shown that a network approach to welfare, which accounts for heterogeneity of local interactions, enables us not only to derive new results and insights on the impact of income redistribution and policy reform but also to link two prominent literatures: on the neutrality result — and its prediction of complete crowding-out of private provision of public goods; and on the transfer paradox — and its close association with international trade and development. We have also shown that given a network structure, its eigenvalues provide us with a useful policy diagnostic: they tell us whether income redistribution is welfare improving or not. As a consequence, in view of the large body of empirical and experimental research in the above mentioned literatures, our findings can provide clear testable predictions for many applications including empirical and experimental work.

A Proofs

Proof of Proposition 3.1 Given two relatively small transfers $\mathbf{t}_1$ and $\mathbf{t}_2$, from Allouch (2015) we have that

$$\mathbf{q}^{\mathbf{t}_1} - \mathbf{q}^{\mathbf{t}_1} = (\mathbf{I} + \mathbf{AG})^{-1}(\mathbf{I} - \mathbf{A})(\mathbf{t}_1 - \mathbf{t}_2),$$

where $\mathbf{A} = \text{diag}(1 - \gamma_i'(\beta_i))_{i \in N}$ for some real numbers $(\beta_i)_{i \in N}$.

We can now calculate the changes in private and public good consumption from the above. We know that $\mathbf{x}^{\mathbf{t}_1} - \mathbf{x}^{\mathbf{t}_2} + \mathbf{q}^{\mathbf{t}_1} - \mathbf{q}^{\mathbf{t}_2} = \mathbf{t}_1 - \mathbf{t}_2$, so

$$\begin{aligned}
\mathbf{x}^{\mathbf{t}_1} - \mathbf{x}^{\mathbf{t}_2} &= \mathbf{t}_1 - \mathbf{t}_2 - (\mathbf{q}^{\mathbf{t}_1} - \mathbf{q}^{\mathbf{t}_2}) = [\mathbf{I} - (\mathbf{I} + \mathbf{AG})^{-1}(\mathbf{I} - \mathbf{A})](\mathbf{t}_1 - \mathbf{t}_2) \\
&= (\mathbf{I} + \mathbf{AG})^{-1}[(\mathbf{I} + \mathbf{AG}) - (\mathbf{I} - \mathbf{A})](\mathbf{t}_1 - \mathbf{t}_2) \\
&= (\mathbf{I} + \mathbf{AG})^{-1}\mathbf{A}(\mathbf{I} + \mathbf{G})(\mathbf{t}_1 - \mathbf{t}_2).
\end{aligned}$$

We also know that $\mathbf{Q}^{\mathbf{t}_1} - \mathbf{Q}^{\mathbf{t}_2} = (\mathbf{I} + \mathbf{G})(\mathbf{q}^{\mathbf{t}_1} - \mathbf{q}^{\mathbf{t}_2})$, so we have that

$$\begin{aligned}
\mathbf{Q}^{\mathbf{t}_1} - \mathbf{Q}^{\mathbf{t}_2} &= (\mathbf{I} + \mathbf{G})(\mathbf{q}^{\mathbf{t}_1} - \mathbf{q}^{\mathbf{t}_2}) = (\mathbf{I} + \mathbf{G})(\mathbf{I} + \mathbf{AG})^{-1}(\mathbf{I} - \mathbf{A})(\mathbf{t}_1 - \mathbf{t}_2) \\
&= (\mathbf{I} + \mathbf{G})(\mathbf{I} + \mathbf{AG})^{-1}[(\mathbf{I} + \mathbf{AG}) - (\mathbf{AG} + \mathbf{A})](\mathbf{t}_1 - \mathbf{t}_2) \\
&= [\mathbf{A}^{-1} - \mathbf{I}](\mathbf{I} + \mathbf{AG})^{-1}\mathbf{A}(\mathbf{I} + \mathbf{G})(\mathbf{t}_1 - \mathbf{t}_2).
\end{aligned}$$

Therefore, it holds that $(x_i^{\mathbf{t}_1}, Q_i^{\mathbf{t}_1}) = (x_i^{\mathbf{t}_2}, Q_i^{\mathbf{t}_2})$, for each i, if and only if $(\mathbf{I} + \mathbf{G})(\mathbf{t}_1 - \mathbf{t}_2) = \mathbf{0}$, which is equivalent to $\hat{\mathbf{t}}_1 = \hat{\mathbf{t}}_2$.

Proof of Proposition 3.2 Observe that $\mathbf{t}$ being a transfer is equivalent to budget balance. Moreover, $\mathbf{t}$ is neutral is equivalent to $\hat{\mathbf{t}} = \mathbf{0}$. This is also equivalent to $(\mathbf{I} + \mathbf{G})\mathbf{t} = 0$, which is also equivalent to $\mathbf{t}$ being an eigenvector to the -1 eigenvalue.

Proof of Proposition 3.3 Following from the definition of main eigenvalues, if -1 is a main eigenvalue, we can choose the corresponding eigenvectors in such a way that, at most, one of them is not orthogonal to $\mathbf{1}$.

Proof of Proposition 4.1 Since we normalised prices to $(1, 1)$ and assumed all consumers are contributors, without loss of generality it holds that

$$\nabla_x \mathbf{u} = \Big(\frac{\partial u_i}{\partial x_i}\Big)_{i \in N} = \nabla_Q \mathbf{u} = \Big(\frac{\partial u_i}{\partial Q_i}\Big)_{i \in N} = \mathbf{1}.$$

Next, we can use Proposition 3.1 and a Taylor approximation such that

$$\begin{aligned}
\Delta \mathbf{v}(\mathbf{t}) &\approx \nabla_x \mathbf{u}(\mathbf{x}_i^{\mathbf{t}} - \mathbf{x}_i^*) + \nabla_Q \mathbf{u}(\mathbf{Q}_i^{\mathbf{t}} - \mathbf{Q}_i^*) \\
&= (\mathbf{I} + \mathbf{AG})^{-1}\mathbf{A}(\mathbf{I} + \mathbf{G})\mathbf{t} + [\mathbf{A}^{-1} - \mathbf{I}](\mathbf{I} + \mathbf{AG})^{-1}\mathbf{A}(\mathbf{I} + \mathbf{G})\mathbf{t} \\
&= \mathbf{A}^{-1}(\mathbf{I} + \mathbf{AG})^{-1}\mathbf{A}(\mathbf{I} + \mathbf{G})\mathbf{t}.
\end{aligned}$$

Let $\mathcal{V}$ be the space of utility changes that can be achieved with local transfers:

$$\mathcal{V} = \{\Delta\mathbf{v}(\mathbf{t}) \in \mathbb{R}^n \mid \Delta\mathbf{v}(\mathbf{t}) = \mathbf{A}^{-1}(\mathbf{I}+\mathbf{AG})^{-1}\mathbf{A}(\mathbf{I}+\mathbf{G})\mathbf{t} \text{ for a local transfer } \mathbf{t}\}.$$

Therefore, given $\mathcal{V}$ is a linear subspace, it follows from Corollary $3'$ of Ben-Israel (1964) that there are two mutually exclusive possibilities, (a) and (b):

$$\begin{array}{ll}(a) & \mathcal{V} \cap \mathbb{R}^n_+ \neq \{0\} \iff \mathcal{V}^\perp \cap \mathbb{R}^n_{++} = \varnothing. \\ (b) & \mathcal{V} \cap \mathbb{R}^n_+ = \{0\} \iff \mathcal{V}^\perp \cap \mathbb{R}^n_{++} \neq \varnothing.\end{array}$$

If (a) then there exist weakly Pareto-improving transfers in the subspace $\mathcal{V}$.

If (b) then weakly Pareto-improving transfers do not exist, and $\mathcal{V}^\perp$ contains strictly positive **r** such that $\mathbf{r}\cdot\Delta\mathbf{v}(\mathbf{t}) = 0$ for any transfer $\mathbf{t}$ and thereby the initial private provision equilibrium is welfare-optimal.

Proof of Proposition 4.2 Suppose there exists no Pareto improvement. Then there exists a vector (of Motzkin weights) $\mathbf{r} \in \mathbb{R}^n_{++}$ such that

$$\mathbf{r}\mathbf{A}^{-1}(\mathbf{I}+\mathbf{AG})^{-1}\mathbf{A}(\mathbf{I}+\mathbf{G}) = \mathbf{1}.$$

Let $\mathbf{s}$ be an eigenvector to the -1 eigenvalue such that $\mathbf{1}\cdot\mathbf{s} \neq 0$. Then it holds that

$$0 = \mathbf{r}\mathbf{A}^{-1}(\mathbf{I}+\mathbf{AG})^{-1}\mathbf{A}(\mathbf{I}+\mathbf{G})\mathbf{s} = \mathbf{1}\cdot\mathbf{s} \neq 0,$$

which is a contradiction.

Proof of Proposition 4.3 Since the indirect utility for Cobb–Douglas preferences is linear, we can simply extend our analysis to relatively small transfers and the Taylor approximation in the proof of Proposition 4.1 holds exactly. Indeed, it follows that condition (b) holds in the case of Cobb–Douglas preferences if and only if there exists a vector (of Motzkin weights) $\mathbf{r} \in \mathbb{R}^n_{++}$ such that $\mathbf{r}(\mathbf{I}+a\mathbf{G})^{-1}(\mathbf{I}+\mathbf{G}) = \mathbf{1}$, which is equivalent to $(\mathbf{I}+a\mathbf{G})^{-1}(\mathbf{I}+\mathbf{G})\mathbf{r} = \mathbf{1}$, since all matrices are symmetric. This is equivalent to $(\mathbf{I}+\mathbf{G})\mathbf{r} = \mathbf{1}(\mathbf{I}+a\mathbf{G})$, which is also equivalent to $\hat{\mathbf{r}} = \mathbf{1}+a\mathbf{d}$.

Proof of Proposition 5.1 From the proof of Proposition 4.1 it follows that

$$\Delta\mathbf{v}(\mathbf{t}) = (\mathbf{I}+a\mathbf{G})^{-1}(\mathbf{I}+\mathbf{G})\mathbf{t}.$$

Therefore, if the transfer $\mathbf{t}$ is an eigenvector to the eigenvalue λ, then it holds that

$$\Delta\mathbf{v}(\mathbf{t}) = \kappa_\lambda\ (\lambda+1)\ \mathbf{t} = \kappa_\lambda\ \hat{\mathbf{t}},$$

where $\kappa_\lambda = \frac{1}{1+a\lambda}$. Having assumed network normality, we note that this is equivalent to $a \in]0, -\frac{1}{\lambda_{\min}(\mathbf{G})}[$ in the Cobb–Douglas preferences case, which ensures that κ_λ is positive.

Proof of Proposition 5.2 If the network is regular, then it has exactly one main eigenvalue, which is the highest eigenvalue. This implies that the eigenvectors to all eigenvalues except the highest eigenvalue are transfers. Complete networks are the only networks such that the lowest eigenvalue is not below -1 and the second eigenvalue is not above -1. Therefore the eigenvector to the second eigenvalue is a transfer with a normal welfare impact and the eigenvector to the lowest eigenvalue is a transfer with paradoxical welfare impact (strong transfer paradox).

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